Fano switching

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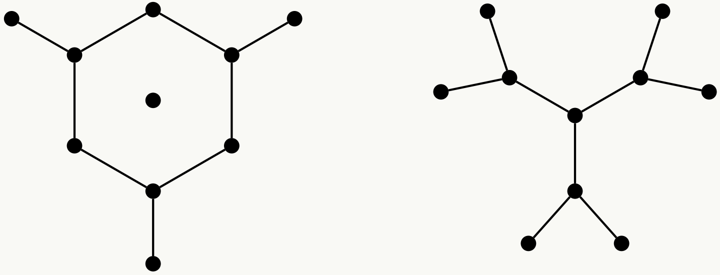
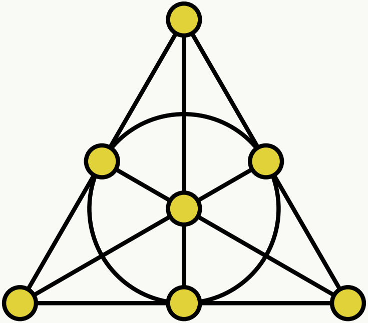
Based on joint work with
Aida Abiad (TU/e), Nils Van de Berg (TU/e) and Ferdinand Ihringer (SUSTech)

Cospectral mates



Both graphs have spectrum $\left\{(-\sqrt{5})^1, (-\sqrt{2})^2, (0)^3, (\sqrt{2})^2, (\sqrt{5})^1\right\}$.

The Fano plane



$\mathrm{PG}(2,2)$

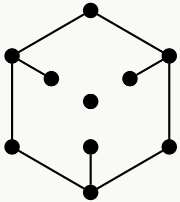
Fano switching

Theorem

Let Γ be a graph with a subgraph C of size 7 whose vertices are identified as points of the Fano plane, such that:

- *C is empty or complete.*
- *Every vertex $v \notin C$ has 0, 3, 4 or 7 neighbours in C.*
  - *If v has 3 neighbours in C, they form a line.*
  - *If v has 4 neighbours in C, they form the complement of a line.*

Let π be a permutation of the lines. For every $v \notin C$ that is (non)adjacent to the vertices of a line ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The obtained graph is cospectral with Γ.



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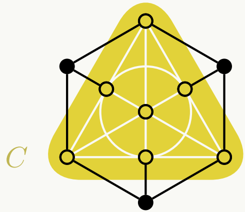

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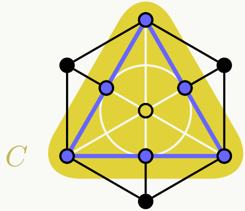

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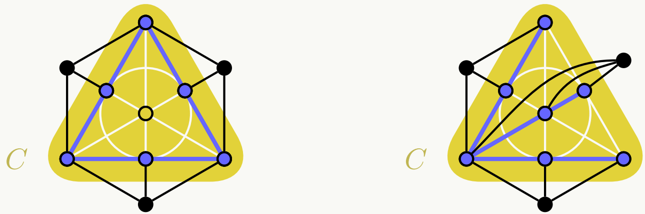

Fano switching

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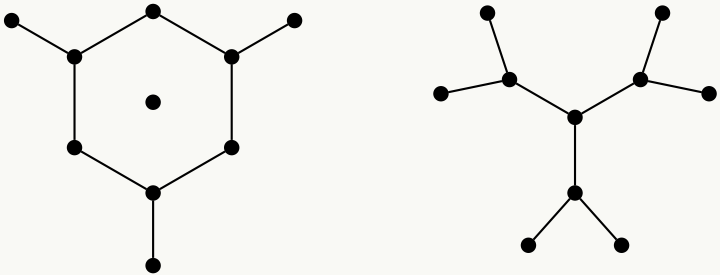
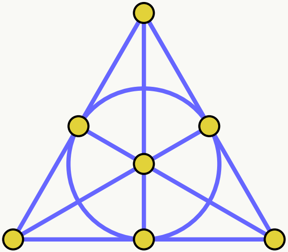


Why does it work?



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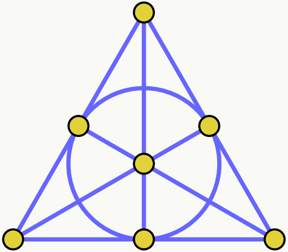

$$N = \begin{array}{c} \\ \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ \ell_6 \\ \ell_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

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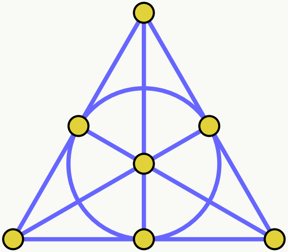

$$N^{\pi} = \begin{array}{c} \\ \ell_1^{\pi} \\ \ell_2^{\pi} \\ \ell_3^{\pi} \\ \ell_4^{\pi} \\ \ell_5^{\pi} \\ \ell_6^{\pi} \\ \ell_7^{\pi} \end{array} \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

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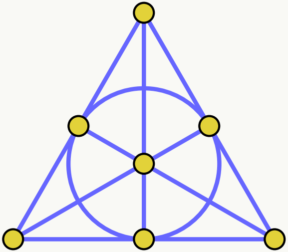

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$R = \frac{1}{2}\left(N^T N^{\pi} - J\right)$ and $Q = \begin{pmatrix} R & O \\ O & I \end{pmatrix} \implies Q^T A Q = A'$

More general

Theorem

Let Γ be a graph with a subgraph C of size 7 whose vertices are identified as points of the Fano plane, such that:

- *C is empty or complete.*
- *Every vertex $v \notin C$ has 0, 3, 4 or 7 neighbours in C.*
  - *If v has 3 neighbours in C, they form a line.*
  - *If v has 4 neighbours in C, they form the complement of a line.*

Let π be a permutation of the lines. For every $v \notin C$ that is (non)adjacent to the vertices of a line ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The obtained graph is cospectral with Γ.

More general

Theorem

Let Γ be a graph with a subgraph C of size q^2+q+1 whose vertices are identified as points of a projective plane of order q, such that:

- *C is empty or complete.*
- *Every vertex $v \notin C$ has 0, $q+1$, q^2 or q^2+q+1 neighbours in C.*
  - *If v has $q+1$ neighbours in C, they form a line.*
  - *If v has q^2 neighbours in C, they form the complement of a line.*

Let π be a permutation of the lines. For every $v \notin C$ that is (non)adjacent to the vertices of a line ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The obtained graph is cospectral with Γ.

- Projective plane switching

More general

Theorem

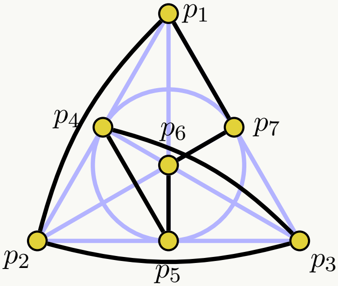
Let Γ be a graph with a subgraph C of size q^2+q+1 whose vertices are identified as points of a projective plane of order q, such that:

- *C is ~~empty or complete~~ **a Singer cycle**.*
- *Every vertex $v \notin C$ has 0, $q+1$, q^2 or q^2+q+1 neighbours in C.*
  - *If v has $q+1$ neighbours in C, they form a line.*
  - *If v has q^2 neighbours in C, they form the complement of a line.*

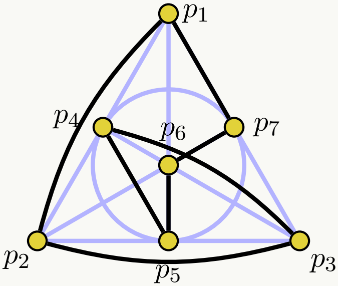
*Let π be **a bijection between the lines and some "new lines that respect the cyclic structure"**. For every $v \notin C$ that is (non)adjacent to the vertices of a line ℓ, make it (non)adjacent to the vertices of $\pi(\ell)$. The obtained graph is cospectral with Γ.*

- Projective plane switching
- Singer cycles

Singer cycle

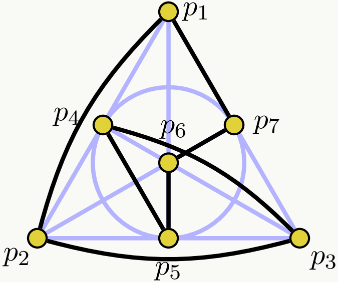


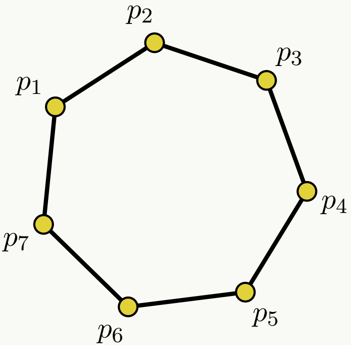
Singer cycle



	p_1	p_2	p_3	p_4	p_5	p_6	p_7
ℓ_1	1	1	0	1	0	0	0
ℓ_2	0	1	1	0	1	0	0
ℓ_3	0	0	1	1	0	1	0
ℓ_4	0	0	0	1	1	0	1
ℓ_5	1	0	0	0	1	1	0
ℓ_6	0	1	0	0	0	1	1
ℓ_7	1	0	1	0	0	0	1

Singer cycle





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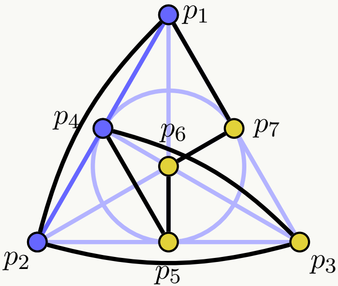

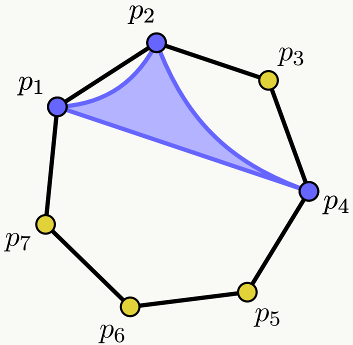

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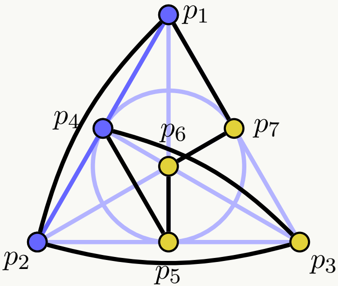

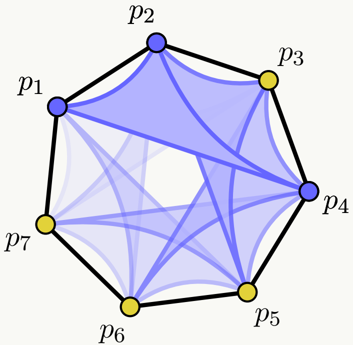

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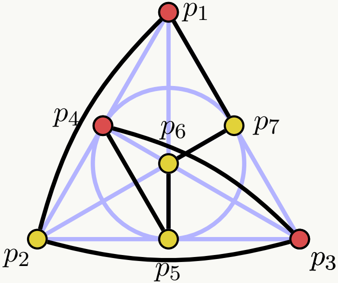

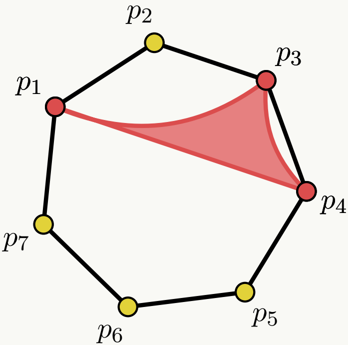

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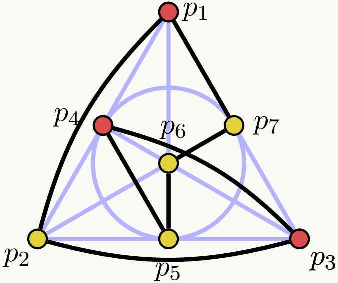

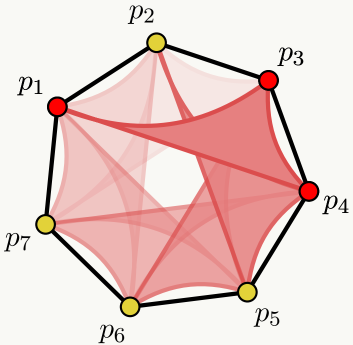

Singer cycle

Theorem

Let Γ be a graph with a subgraph $C \cong C_7$, labelled from 1 to 7. Define $\ell_i = \{i, i{+}1, i{+}3\}$ and $\mathcal{O}_i = \{i, i{+}2, i{+}3\}$ such that:

- *Every vertex $v \notin C$ has 0, 3, 4 or 7 neighbours in C.*
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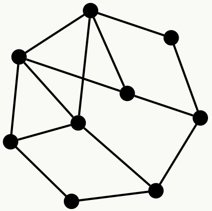

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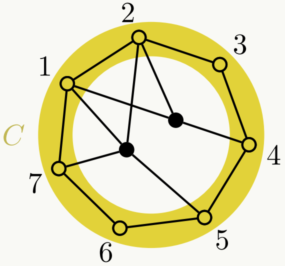

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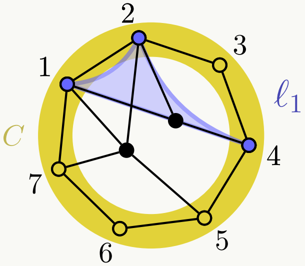

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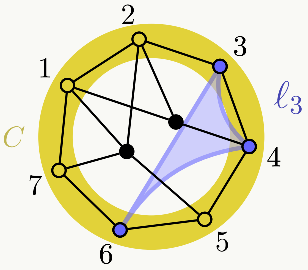

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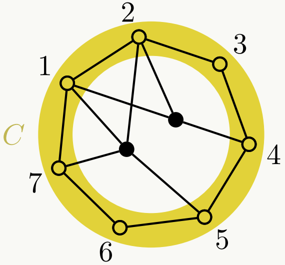

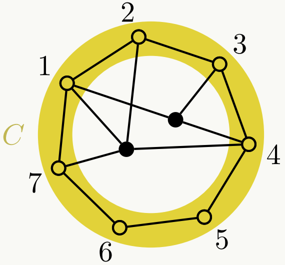

Thank you for listening!

A. Abiad, N. Van de Berg and R. Simoens,
Switching methods of level 2 for the construction of cospectral graphs,
arXiv:2410.07948.